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Diagonalization of matrices over H^∞

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Homage to the memory of F. Riesz (1880—1956)

By H^∞ we mean the Banach algebra of bounded holomorphic functions $u(\lambda)$ on the disc $|\lambda|<1$, with the sup-norm $\|u\|_\infty$. For the relevant fundamental notions and facts (inner functions and their canonical representation, inner factor of a non-zero $u\in H^\infty$, largest common inner divisor $\bigwedge_\alpha u_\alpha$, and least common inner multiple $\bigvee_\alpha u_\alpha$ (if it exists), of a family $\{u_\alpha\}$ of inner functions, etc.) we refer e.g. to [4], Chapter III. It is convenient to define $\bigwedge_\alpha v_\alpha$ for *any* family $\{v_\alpha\}$ of elements of H^∞: this is the largest common inner divisor of the v_α whenever not all v_α are zero, and 0 otherwise. Note that the operations $\wedge$, $\vee$ are defined up to constant factors of modulus 1.

Matrices over H^∞ naturally occur in the theory of unitary equivalence, similarity, or quasi-similarity models of certain types of operators on Hilbert space, as made clear e.g. by the investigations of SZ.-NAGY—FOIAŞ [4], [5], [7]. It was in particular the paper [5] first establishing a Jordan model theory for operators of class C_0 which pointed out the need for a diagonalization theory of matrices over H^∞. This task was achieved, for finite rectangular matrices over H^∞, by NORDGREN [3]. The classical equivalence theory cannot be applied here since the algebra H^∞ does not possess all properties required. However, by introducing a convenient generalization of the notion of equivalence for matrices, called *quasi-equivalence,* Nordgren was able to extend the classical results to this case. SZŰCS [10] gave an analysis of the abstract algebraic background of Nordgren's theory.

The results of [3] were applied in [1], [2], [8] to obtain Jordan models for some classes of operators on Hilbert space, namely to contractions T with finite defect indices and of class C_0 (i.e. such that $T^{*n}\to 0$).

The aim of the present paper is to extend the Nordgren diagonalization theory. The key to this extension is the Main Lemma (Sec.2) which establishes a remarkable property of H^∞. It can be applied to solve the diagonalization problem for finite

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and semi-finite matrices over H^∞ as well, and also to get some insight into the case of (doubly) infinite matrices (Sec.3). The full solution of the problem of infinite matrices would, however, require further study because of the convergence difficulties which there arise.

The concluding Sec. 4 indicates how the matrix diagonalization results can be applied to obtain a Jordan model of operators $T \in C_{\cdot 0}$ with at least one finite defect index, thus generalizing the results of [8] and [2].

1. Preliminaries

For convenience of reference we begin with some more or less known lemmas.

Lemma 1. *Let ω be an inner function and let $p_\alpha, q_\alpha (\alpha \in A)$ be inner divisors of ω such that $p_\alpha \cdot q_\alpha = \omega$ for each $\alpha \in A$. Then,*

$$\bigwedge_\alpha p_\alpha \cdot \bigvee_\alpha q_\alpha = \omega.$$

Poof. $q^\vee = \bigvee_\alpha q_\alpha$ is divisible by each q_α; hence there exist inner functions v_α such that $q^\vee p_\alpha = \omega v_\alpha$. Since $q^\vee$ is a divisor of ω, we have $p_\alpha = (\omega/q^\vee) \cdot v_\alpha$ for all α. Then $\omega/q^\vee$ is a divisor of $p^\wedge = \bigwedge_\alpha p_\alpha$ also. Therefore, we have

$$\omega/p^\wedge \mid q^\vee.$$

On the other hand, we have $\omega/p^\wedge = (\omega/p_\beta)(p_\beta/p^\wedge) = q_\beta \cdot (p_\beta/p^\wedge)$, and hence $q_\beta \mid (\omega/p^\wedge)$ for every $\beta \in A$, and therefore,

$$q^\vee \mid \omega/p^\wedge.$$

The two relations yield the result we wished to prove.

Corollary. *Under the hypotheses of Lemma* 1 *we have*

$$\bigvee_\alpha q_\alpha = \omega \quad \textit{if and only if} \quad \bigwedge_\alpha p_\alpha = 1.$$

Lemma 2. (M. Sherman, *cf.* [6]) *Let $f_1, f_2 \in H^\infty$ and let ω be an inner function. Then for every complex number t, with the exception of at most countable many values, we have*

$$\omega \wedge (f_1 + t f_2) = \omega \wedge f_1 \wedge f_2.$$

Proof. Let $g_1, g_2 \in H^\infty$ be any pair linearly equivalent (with constant coefficients) to the pair f_1, f_2. Then $g_1 \wedge g_2 = f_1 \wedge f_2$, and hence

$$\omega \wedge g_1 \wedge g_2 = \omega \wedge f_1 \wedge f_2. \tag{1.1}$$

Applying Lemma 1 to the inner function ω and to its inner divisors $p_\alpha = \omega \wedge g_\alpha$ and $q_\alpha = \omega / p_\alpha$ $(\alpha = 1, 2)$ we get, taking account of (1.1),

$$q_1 \vee q_2 = \frac{\omega}{p_1 \wedge p_2} = \frac{\omega}{\omega \wedge g_1 \wedge g_2} = \frac{\omega}{\omega \wedge f_1 \wedge f_2} (= \Omega).$$

By the corollary of Lemma 1, applied with this Ω in place of ω, we obtain that

$$\Omega / q_1 \wedge \Omega / q_2 = 1.$$

Consider now the one parameter family of functions $h_t = f_1 + t f_2$. For $t_1 \neq t_2$ the pair h_{t_1}, h_{t_2} is linearly equivalent to the pair f_{t_1}, f_{t_2}. Hence, the family of functions

$$\Omega \bigg/ \frac{\omega}{\omega \wedge h_t} = \frac{\omega \wedge h_t}{\omega \wedge f_1 \wedge f_2} \quad (t \text{ complex parameter})$$

consists of pairwise prime inner divisors of Ω.

Now, it follows from the canonical representation of the inner function Ω (by its zeros in the unit disc and the corresponding singular measure on the unit circle) that no family of pairwise prime inner divisors of Ω can contain more than countably many non-constant elements. Thus, for all values of the parameter t, with the possible exception of a countable set of values, we have

$$\omega \wedge h_t = \omega \wedge f_1 \wedge f_2.$$

This concludes the proof.

Lemma 3. *Let $\mathcal{I}$ be a family of inner functions such that*

(i) $u_1, u_2 \in \mathcal{I}$ *imply* $u_1 \vee u_2 \in \mathcal{I}$,

(ii) $\inf_{u \in \mathcal{I}} |u(\lambda_0)| > 0$ *for some point* λ_0, $|\lambda_0| < 1$.

Then $u^\vee = \bigvee_{u \in \mathcal{I}} u$ *exists and every sequence* u_n *minimizing* $|u(\lambda_0)|$ *has a subsequence converging to* $u^\vee$ *in the unit disc* $|\lambda| < 1$.

For a proof, based on the Vitali—Montel theorem, *cf.* [6] or [7], Lemma 1.

2. Main Lemma

The following lemma on functions in H^∞ is related to a theorem on Hilbert space operators, proved in [7] (Theorem 1). We present here a direct proof, using elements of the proof of the operator theoretic theorem in [7]. (Although we shall only use in this paper the case when $\omega_i = \omega$ for all i, the general case is considered in view of possible further applications.)

Main Lemma. *Let* $f_{ik} \in H^\infty$, $\|f_{ik}\|_\infty \leqq M$ $(i, k = 1, 2, \ldots)$, *and let* ω_i $(i = 1, 2, \ldots)$ *be inner functions. Suppose that*

$$\omega_i \wedge f_{i1} \wedge f_{i2} \wedge \ldots = 1 \quad (i = 1, 2, \ldots). \tag{2.1}$$

Then there exists a numerical sequence $\langle x_2, x_3, \dots\rangle$, *with* $\Sigma|x_k|$ *as small as we wish, such that*

$$(2.2)\qquad \omega_i\wedge(f_{i1}+x_2f_{i2}+x_3f_{i3}+\dots)=1\quad (i=1,2,\dots).$$

Proof. a) Consider the linear transformations

$$r_i: l^1\to H^\infty\quad (i=1,2,\dots)$$

defined for $x=\langle x_1, x_2, \dots\rangle\in l^1$ by

$$r_i x=\sum_1^\infty x_k f_{ik};$$

clearly, $\|r_i x\|_\infty\leqq M\,\|x\|_1$. Denote by R_i the range of r_i in H^∞.

Condition (2.1) is obviously equivalent to

$$\bigwedge_{g\in R_i}(\omega_i\wedge g)=1\quad (i=1,2,\dots)$$

and this in its turn is equivalent, by the corollary of Lemma 1, to

$$(2.3)\qquad \bigvee_{g\in R_i}\frac{\omega_i}{\omega_i\wedge g}=\omega_i\quad (i=1,2,\dots).$$

Choose a point λ_0, $|\lambda_0|<1$, different from the zeros of the functions $\omega_1, \omega_2, \dots$; thus

$$(2.4)\qquad |\omega_i(\lambda_0)|=\mu_i>0;$$

and define

$$(2.5)\qquad \nu_i=\inf_{g\in R_i}\left|\frac{\omega_i}{\omega_i\wedge g}(\lambda_0)\right|\quad (i=1,2,\dots).$$

Clearly, $\nu_i\geqq|\omega_i(\lambda_0)|=\mu_i$; thus the family of functions

$$\mathscr{I}_i=\left\{\frac{\omega_i}{\omega_i\wedge g}: g\in R_i\right\}$$

satisfies condition (ii) in Lemma 3. It also satisfies condition (i). For, if $g_1, g_2\in\mathscr{I}_i$ then by linearity of l^1 and r_i we have $g_1+tg_2\in R_i$ for all values of the complex parameter t. Now, by Lemma 2 we have $\omega_i\wedge(g_1+tg_2)=(\omega_i\wedge g_1)\wedge(\omega_i\wedge g_2)$ for all t with the possible exception of countable many, and for a non-exceptional value of t we have by Lemma 1

$$\frac{\omega_i}{\omega_i\wedge g_1}\vee\frac{\omega_i}{\omega_i\wedge g_2}=\frac{\omega_i}{\omega_i\wedge(g_1+tg_2)};$$

thus condition (i) holds true for each $\mathscr{I}_i$.

Fix i and consider a sequence $\{g_n\}$ minimizing in (2.5); by virtue of Lemma 3 we can choose this sequence even so that

$$(2.6) \qquad \frac{\omega_i}{\omega_i \wedge g_n} \to \bigvee_{g \in R_i} \frac{\omega_i}{\omega_i \wedge g} \quad \text{pointwise in} \quad |\lambda| < 1, \quad \text{as} \quad n \to \infty.$$

By Lemma 1 and by (2.3), this limit equals

$$\omega_i \Big/ \bigwedge_{g \in R_i} (\omega_i \wedge g), \quad \text{i.e.} \quad \omega_i.$$

Thus we have, in particular,

$$(2.7) \qquad \nu_i = \lim_{n \to \infty} \left| \frac{\omega_i}{\omega_i \wedge g_n}(\lambda_0) \right| = |\omega_i(\lambda_0)| = \mu_i \quad \text{for all} \quad i.$$

b) Next we assert that the infimum ν_i in (2.5) is attained for every value of i. Moreover, we assert that there exists an $x = \langle x_1, x_2, \dots \rangle \in l^1$, independent of i, such that, for every i, the infimum ν_i is attained for $g_i = r_i x$, that is,

$$\left| \frac{\omega_i}{\omega_i \wedge r_i x}(\lambda_0) \right| = \nu_i = \mu_i = |\omega_i(\lambda_0)|, \quad |(\omega_i \wedge r_i x)(\lambda_0)| = 1 \quad (i = 1, 2, \dots).$$

By the maximum principle this implies $\omega_i \wedge r_i x = 1$, i.e.

$$(2.8) \qquad \omega_i \wedge (x_1 f_{i1} + x_2 f_{i2} + \dots) = 1 \quad (i = 1, 2, \dots).$$

To prove our assertion suppose the contrary, i.e., that for every $x \in l^1$ we have

$$\left| \frac{\omega_i}{\omega_i \wedge r_i x}(\lambda_0) \right| > \mu_i$$

for at least one subscript i, or equivalently, that l^1 is the union of the subsets

$$\sigma_{ij} = \left\{ x : x \in l^1, \left| \frac{\omega_i}{\omega_i \wedge r_i x}(\lambda_0) \right| \geqq \mu_i + \frac{1}{j} \right\} \quad (i, j = 1, 2, \dots).$$

Let us show that each of these subsets is *closed.*

To this effect consider a sequence of vectors $x_n \in \sigma_{ij}$ (i, j fixed), converging in l^1 to a limit x; then

$$g_n = r_i x_n, \quad g = r_i x \quad \text{satisfy} \quad \|g_n - g\|^\infty \leqq \|r_i\| \, \|x_n - x\|_1 \to 0 \quad (n \to \infty)$$

and therefore we have, in particular,

$$(2.9) \qquad g_n \to g \quad \text{pointwise in} \quad |\lambda| < 1.$$

Passing, if necessary, to a subsequence we can also assume, by virtue of the Vitali—Montel theorem, that

$$(2.10) \qquad \frac{\omega_i}{\omega_i \wedge g_n} \to p, \quad \frac{g_n}{\omega_i \wedge g_n} \to q \quad \text{pointwise for } |\lambda| < 1 \quad \text{as} \quad n \to \infty,$$

where p and q are analytic for $|\lambda|<1$; clearly $\|p\|_\infty \leqq 1$ and $\|q\|_\infty \leqq M \cdot \sup \|x_n\|_1$. Note that, in particular,

(2.11) $$|p(\lambda_0)| = \lim_{n\to\infty} \left| \frac{\omega_i}{\omega_i \wedge g_n}(\lambda_0) \right| \geqq \mu_i + \frac{1}{j}.$$

From (2.9) and (2.10) we infer

$$\frac{\omega_i}{\omega_i \wedge g_n} g_n \to pg, \quad \omega_i \frac{g_n}{\omega_i \wedge g_n} \to \omega_i q \quad \text{pointwise, as} \quad n \to \infty,$$

and hence, $pg = \omega_i q$, $p^\circ g^\circ = \omega_i q^\circ$, where the superscript $^\circ$ indicates inner factor. It follows that $\dfrac{\omega_i}{\omega_i \wedge g}$ is an inner divisor of p°, and hence

(2.12) $$\left| \frac{\omega_i}{\omega_i \wedge g}(\lambda_0) \right| \geqq |p^\circ(\lambda_0)| \geqq |p(\lambda_0)|,$$

because the outer factor $p' = p/p^\circ$ has the same norm $\|\cdot\|_\infty$ as p, thus $|p'(\lambda)| \leqq 1$ for $|\lambda|<1$. From (2.11) and (2.12) we infer that $x \in \sigma_{ij}$: σ_{ij} is closed.

Thus l^1 is the union of the closed subsets σ_{ij} $(i, j = 1, 2, \ldots)$. By virtue of the Baire category theorem, at least one of the sets σ_{ij} must contain a ball

$$\mathscr{B} = \{x : \|x - x_0\| < \varrho\} \quad \text{in} \quad l^1,$$

that is, there exist a subscript i and a number μ_i' greater than μ_i, such that

(2.13) $$\left| \frac{\omega_i}{\omega_i \wedge g}(\lambda_0) \right| \geqq \mu_i' \quad \text{for all} \quad g \in r_i \mathscr{B}.$$

On the other hand, on account of the equality $\nu_i = \mu_i$ we have $\nu_i < \mu_i'$, and therefore there exists $y \in l^1$ such that

(2.14) $$\left| \frac{\omega_i}{\omega_i \wedge h}(\lambda_0) \right| < \mu_i' \quad \text{for} \quad h = r_i y.$$

Set $f_0 = r_i x_0$ and apply Lemma 2 to obtain that there exists t, $0 < t < \varrho/\|y\|_1$, such that

(2.15) $$\frac{\omega_i}{\omega_i \wedge (f_0 + th)} = \frac{\omega_i}{\omega_i \wedge f_0} \vee \frac{\omega_i}{\omega_i \wedge h}.$$

As we have $f_0 + th = r_i(x_0 + ty) \in r_i \mathscr{B}$, the function at the left hand side of (2.15) has, by (2.13), absolute value $\geqq \mu_i'$. The function at the right hand side of (2.15), being an inner multiple of the function $\dfrac{\omega_i}{\omega_i \wedge h}$, is majorized in absolute value by the latter function everywhere in the unit disc; thus by (2.14) the function at the right hand side of (2.15) has at the point λ_0 absolute value $< \mu_i'$.

So we have arrived at a contradiction. This proves our assertion stated at the beginning of part b) of the proof, namely that there exists an $x \in l^1$ satisfying (2.8).

c) In the last step of our proof we shall again refer to the (Sherman) Lemma 2. Let $x = \langle x_1, x_2, \dots \rangle \in l^1$ be any vector for which (2.8) holds, i.e. such that

$$\omega_i \wedge \varphi_i = 1 \quad \text{for} \quad \varphi_i = x_1 f_{i1} + x_2 f_{i2} + \dots \quad (i = 1, 2, \dots).$$

Then by Lemma 2 we also have

$$\omega_i \wedge (\varphi_i + t f_{i1}) = \omega_i \wedge \varphi_i \wedge f_{i1} = 1 \wedge f_{i1} = 1 \quad (i = 1, 2, \dots)$$

for all values of the complex parameter t, with the possible exception of countably many values. Given $\varepsilon > 0$, if we choose t not exceptional for any i, and moreover different from $-x_1$ and sufficiently large, we will have

$$\omega_i \wedge (f_{i1} + x_2' f_{i2} + x_3' f_{i3} + \dots) = 1 \quad (i = 1, 2, \dots),$$

with $x_k' = x_k/(x_1 + t)$ and $\sum_2^\infty |x_k'| < \varepsilon$.

This completes the proof of the Main Lemma.

When referring to the Main Lemma we shall mean its following direct corollary:

Let a_{ik} be a (finite, semi-finite, or infinite) rectangular matrix over H^∞, with $\|a_{ik}\|_\infty \leqq M$, and let ω be an inner function. Then there exists a numerical sequence $\langle x_2, x_3, \dots \rangle$, with $\sum |x_k|$ as small as we wish, such that, for every value of i, we have

$$a_{i1} + x_2 a_{i2} + x_3 a_{i3} + \dots = h_i \cdot (a_{i1} \wedge a_{i2} \wedge a_{i3} \wedge \dots),$$

where $h_i \in H^\infty$, $h_i \wedge \omega = 1$.

3. Quasi-equivalence and diagonalization of matrices over H^∞

1. Let $\mathcal{M}(n, m)$ $(1 \leqq n \leqq \infty,\ 1 \leqq m \leqq \infty)$ be the set of $n \times m$ matrices $A = [a_{ik}]$ over H^∞, for which

$$\sum_i \Big| \sum_k \xi_k a_{ik}(\lambda) \Big|^2 \leqq M^2 \sum_k |\xi_k|^2 \quad (M \geqq 0) \tag{3.1}$$

holds for $|\lambda| < 1$ and for any square-summable sequence of complex numbers ξ_k, i.e. whose values $A(\lambda)$ $(|\lambda| < 1)$ are operators from (complex euclidean) m-space E_m into n-space E_n, bounded by the constant M,

$$\|A\|_\infty = \sup_{|\lambda| < 1} \|A(\lambda)\| \leqq M.$$

By $\mathcal{N}(n)$ $(1 \leqq n \leqq \infty)$ we denote the set of matrices $X = X(\lambda)$ in $\mathcal{M}(n, n)$ for which $X(\lambda)^{-1}$ exists $(|\lambda| < 1)$ and also belongs to $\mathcal{M}(n, n)$.

Finally, for a given inner function ω we denote by $\mathcal{N}_\omega(n)$ the set of matrices $X \in \mathcal{M}(n, n)$ which have a scalar multiple φ prime to ω, that is, for which there exists $X^a \in \mathcal{M}(n, n)$ such that

$$X^a X = XX^a = \varphi \cdot I_n, \quad \varphi \in H^\infty, \quad \varphi \not\equiv 0, \quad \varphi \wedge \omega = 1$$

(I_n is the unit matrix of order n).

It is clear that $\mathcal{N}(n) \subset \mathcal{N}_\omega(n)$, and that a finite product of elements of $\mathcal{N}_\omega(n)$ also belongs to $\mathcal{N}_\omega(n)$.

Let $A, B \in \mathcal{M}(n, m)$. We call A, B *equivalent* if there exist matrices $X \in \mathcal{N}(n)$, $Y \in \mathcal{N}(m)$ such that

$$XA = BY, \tag{3.2}$$

and we call them *ω-equivalent* if there exist $X \in \mathcal{N}_\omega(n)$, $Y \in \mathcal{N}_\omega(m)$ satisfying (3.2).

Equivalence implies ω-equivalence, but not conversely. Both are *symmetric*. This is obvious for equivalence, while for ω-equivalence it can be shown as follows: If $X^a X = XX^a = \varphi \cdot I_n$, $Y^a Y = YY^a = \psi \cdot I_m$, $\varphi \wedge \omega = 1$, $\psi \wedge \omega = 1$, then (3.2) implies:

$$A \cdot \varphi Y^a = \varphi A Y^a = X^a X A Y^a = X^a B Y Y^a = X^a B \psi I_m = \psi X^a \cdot B,$$

where $\varphi Y^a \in \mathcal{N}_\omega(m)$ and $\psi X^a \in \mathcal{N}_\omega(n)$ because

$$\varphi Y^a \cdot Y = \varphi \psi I_m = Y \cdot \varphi Y^a, \quad \varphi \psi \wedge \omega = 1,$$

and similarly for ψX^a. — Clearly, both kinds of equivalence are *transitive*.

In case A, B are ω-equivalent for *every* inner ω, they are called *quasi-equivalent*.

These concepts were introduced by NORDGREN [3]; see also SZŰCS [10].

"Determinant divisors" $\mathcal{D}_k$ and "invariant factors" $\mathcal{E}_k$ of a matrix $A \in \mathcal{M}(n, m)$ are defined, for all (finite) integers k, $1 \leq k \leq \min\{n, m\}$, as in the classical case, namely:

$\mathcal{D}_k = \wedge \det A^{(k)}$, where $A^{(k)}$ runs over the set of all square submatrices of A of order k (thus $\mathcal{D}_k = 0$ iff all these submatrices have determinant 0, and $\mathcal{D}_k = 0$ implies $\mathcal{D}_{k+1} = 0$);

$\mathcal{E}_k = \mathcal{D}_k / \mathcal{D}_{k-1}$, with the conventions $\mathcal{D}_0 = 1$ and $\mathcal{E}_k = 0$ if $\mathcal{D}_{k-1} = 0$.

Lemma 4. *If $A, B \in \mathcal{M}(n, m)$ are ω-equivalent, then*

$$\mathcal{D}_k(A) | \alpha_k \mathcal{D}_k(B), \quad \mathcal{D}_k(B) | \beta_k \mathcal{D}_k(A) \quad (k = 1, 2, \ldots), \tag{3.3}$$

where α_k, β_k are inner functions prime to ω. If A, B are even quasi-equivalent, then

$$\mathcal{D}_k(A) = \mathcal{D}_k(B) \quad (k = 1, 2, \ldots). \tag{3.4}$$

Proof. Suppose $X \in \mathcal{N}_\omega(n)$ and $Y \in \mathcal{N}_\omega(m)$ satisfy (3.2). If φ and ψ are their corresponding scalar multiples, prime to ω, then we deduce from (3.2) that

$$X^a B Y = \varphi \cdot A, \quad X A Y^a = \psi \cdot B. \tag{3.5}$$

As the Cauchy—Binet multiplication rule for minors extends to the present case, we get from (3.5), first, that $\mathcal{D}_k(A)=0$ iff $\mathcal{D}_k(B)=0$. Next, if $\mathcal{D}_k(A)$ and $\mathcal{D}_k(B)$ are non-zero, and therefore inner functions, we deduce that

$$\mathcal{D}_k(B)\,|\,\varphi^k\mathcal{D}_k(A) \quad \text{and} \quad \mathcal{D}_k(A)\,|\,\psi^k\mathcal{D}_k(B), \tag{3.6}$$ [1]

and we have only to observe that φ^k and ψ^k are also prime to ω.

If A, B are quasi-equivalent and if for fixed k such that $\mathcal{D}_k(A)$ and $\mathcal{D}_k(B)$ are non-zero we choose $\omega=\mathcal{D}_k(A)\,\mathcal{D}_k(B)$, then φ^k and ψ^k are prime to $\mathcal{D}_k(A)$ and $\mathcal{D}_k(B)$, so that (3.6) implies $\mathcal{D}_k(B)|\mathcal{D}_k(A)$ and $\mathcal{D}_k(A)|\mathcal{D}_k(B)$, i.e. (3.4).

2. For later use we introduce the following notations:

Let $u=\langle 0, u_1, u_2, \ldots\rangle$ be a sequence of length n (finite or infinite) of functions in H^∞ satisfying

$$\|u\|_\infty = \sup_{|\lambda|<1} \big(\sum_k |u_k(\lambda)|^2\big)^{1/2} < \infty$$

and let $C(u)$ and $R(u)$ respectively denote the square matrices whose first column or the first row is given by this sequence and all other entries are 0. These matrices obviously belong to $\mathcal{M}(n, m)$, with $\|C(u)\|_\infty=\|R(u)\|_\infty=\|u\|_\infty$; moreover the matrices $I\pm C(u)$, $I\pm R(u)$ belong to $\mathcal{N}(n)$ because

$$(I\pm C(u))(I\mp C(u)) = I, \quad (I\pm R(u))(I\mp R(u)) = I.$$

Every diagonal (square) matrix $D=\operatorname{diag}(w_1, w_2, \ldots)$ of order n whose diagonal entries are inner functions and have a common inner multiple w, belongs to $\mathcal{N}_\omega(n)$ for every ω such that $w\wedge\omega=1$; indeed,

$$\|D\|_\infty = 1 \quad \text{and} \quad D^a D = DD^a = wI, \quad \text{where} \quad D^a = \operatorname{diag}\left(\frac{w}{w_1}, \frac{w}{w_2}, \ldots\right).$$

Finally, observe that if A_0, A_1 are ω-equivalent to A_0', A_1', then $A=A_0\oplus A_1$ is ω-equivalent to $A'=A_0'\oplus A_1'$. Indeed, if X_0, Y_0 and X_1, Y_1 are operators for A_0, A_0' and A_1, A_1', with the respective scalar multiples φ_0, ψ_0, and φ_1, ψ_1, prime to ω, then $X=X_0\oplus X_1$, $Y=Y_0\oplus Y_1$ will correspond to the pair A, A', and setting

$$X^a = \varphi_1 X_0^a \oplus \varphi_0 X_1^a, \quad Y^a = \psi_1 Y_0^a \oplus \psi_0 Y_1^a$$

we see that X, Y have the scalar multiples $\varphi_0\cdot\varphi_1$, and $\psi_0\cdot\psi_1$, respectively, which are also prime to ω.

3. We are now able to prove:

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[1]) Here we use the fact that if $\{u_\alpha\}$ is a system of inner functions and f is a function in L^∞ such that $fu_\alpha\in H^\infty$ for all α then $f\cdot\bigwedge_\alpha u_\alpha\in H^\infty$; *cf.* Proposition III. 1. 5 in [4]. This fact implies, namely, that if w is inner, if v is in H^∞, and if $w|vu_\alpha$ for all α, then $w\big|\big(v\cdot\bigwedge_\alpha u_\alpha\big)$ (set $f=\bar{w}v$).

Theorem 1. *Let $A=[a_{ik}]\in\mathcal{M}(n,m)$, $1\leqq n\leqq\infty$, $1\leqq m\leqq\infty$, and let r be an integer, $1\leqq r\leqq\min\{n,m\}$. Then, for any given inner function ω, A is ω-equivalent to a matrix of the form*

$$\operatorname{diag}[\mathcal{E}_1(A),\ldots,\mathcal{E}_r(A),A_r]$$

where $A_r\in\mathcal{M}(n_r',m_r)$ $(r+n_r=n,\ r+m_r=m)$, and we have

$$\mathcal{E}_1(A)\,|\,\mathcal{E}_2(A)\,|\ldots|\,\mathcal{E}_r(A)\,|\,A_r.$$

Proof. The case $A=0$ being trivial we can assume $A\neq 0$ so that $\mathcal{D}_1(A)$ is an inner function. From (3.1) it follows, in particular, that $\|a_{ik}\|_\infty\leqq\|A\|_\infty$ $(=M)$.

Denote by ω_r the product of the given inner function ω by the non-zero (and hence, inner) terms of the sequence $\mathcal{D}_1(A),\ldots,\mathcal{D}_r(A)$. Then any $h\in H^\infty$ prime to ω_r is prime to ω as well as to each of these determinant divisors of A.

By virtue of the Main Lemma there exists a numerical sequence $\langle x_1,x_2,\ldots\rangle$ of length m, with $x_1=1$ and $\sum_2^m|x_k|$ as small as we wish, such that

$$(3.7)\qquad (a_i=)\sum_{k=1}^m x_k a_{ik}=h_i\cdot\bigwedge_{k=1}^m a_{ik},\quad h_i\in H^\infty,\quad h_i\wedge\omega_r=1\quad(i=1,2,\ldots,n).$$

Then

$$\sum_i|a_i|^2=\sum_i\Big|\sum_k x_k a_{ik}\Big|^2\leqq M^2\sum_k|x_k|^2\leqq M'^2$$

for some M' (as close to M as we wish) and for all λ, $|\lambda|<1$. Hence, $\|a_i\|_\infty\leqq M'$.

Appling the Main Lemma again we can choose a numerical sequence $\langle y_1,y_2,\ldots\rangle$ of length n, with $y_1=1$ and $\sum_2^m|y_i|$ as small as we wish, such that

$$\sum_{i=1}^n y_i a_i=h'\cdot\bigwedge_{i=1}^n a_i,\quad h'\in H^\infty,\quad h'\wedge\omega_r=1.$$

Observe that there is an inner function h'' such that $h''\wedge\omega_r=1$ and

$$\bigwedge_i a_i=\bigwedge_i\Big(h_i\cdot\bigwedge_k a_{ik}\Big)=h''\cdot\bigwedge_{i,k}a_{ik}.\ ^{2)}$$

We have, therefore,

$$(3.8)\qquad \sum_i y_i a_i=h\cdot\mathcal{D}_1(A),\quad\text{where}\quad h=h'h'',\quad h\wedge\omega_r=1.$$

---

2) Set $b_i=\bigwedge_k a_{ik}$ and $b=\bigwedge_i b_i$: then $b=\mathcal{D}_1(A)$ and $\bigwedge_i(b_i/b)=1$. We have

$$\bigwedge_i a_i=\bigwedge_i(h_ib_i)=\left(\bigwedge_i\left(h_i\frac{b_i}{b}\right)\right)\cdot b\equiv h''\cdot b.$$

Since $h_i\wedge\omega_r=1$, we have

$$h''\wedge\omega_r=\bigwedge_i\left(\left(h_i\frac{b_i}{b}\right)\wedge\omega_r\right)=\bigwedge_i\left(\frac{b_i}{b}\wedge\omega_r\right)=1.$$

The author is indepted to Prof. T. Ando for this proof and also for some other useful remarks he has made when reading the manuscript.

Form the matrices $C_m(x)$ and $R_n(y)$ associated with the sequences $x=\langle 0, x_2, x_3, \dots\rangle$ and $y=\langle 0, y_2, y_3, \dots\rangle$ according to Subsection 2. From (3.7) and (3.8) we deduce that the matrix

$$A' = [a'_{ik}] = (I_n+R_n(y))A(I_m+C_m(x)) \tag{3.9}$$

has the leading entry $a'_{11}=h\cdot\mathscr{D}_1(A)$, while $a'_{ik}=a_{ik}$ for $i, k\geqq 2$. As $I+R_n$ and $I+C_m$ are invertible, A is equivalent to A', and therefore, by Lemma 4,

$$\mathscr{D}_k(A) = \mathscr{D}_k(A') \quad \text{for every} \quad k,$$

in particular $\mathscr{D}_1(A)|A'$.

Now, we set

$$A'' = \begin{bmatrix} \mathscr{D}_1(A) & a'_{12} & a'_{13} & \dots \\ a'_{21} & ha_{22} & ha_{32} & \dots \\ a'_{31} & ha_{32} & ha_{33} & \dots \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

and observe that

$$A'\cdot \operatorname{diag}(1, h, h, \dots)_m = \operatorname{diag}(h, 1, 1, \dots)_n\cdot A'';$$

as a consequence, A' is ω_r-equivalent to A''.

Next, form the matrices $C_n(u)$ and $R_m(v)$ associated with the sequences $u=\langle 0, u_2, u_3, \dots\rangle$ and $v=\langle 0, v_2, v_3, \dots\rangle$, where $u_i=a'_{i1}/\mathscr{D}_1(A)$ and $v_k=a'_{1k}/\mathscr{D}_1(A)$. Because $\mathscr{D}_1(A)$ is an inner function, we have $\|u\|_\infty=\|a'_{.1}\|_\infty$ and $\|v\|_\infty=\|a'_{1.}\|_\infty$, where $a'_{.1}=\langle 0, a'_{21}, a'_{31}, \dots\rangle$ and $a'_{1.}=\langle 0, a'_{12}, a'_{13}, \dots\rangle$. Hence, the matrices $I_n-C_n(u)$ and $I_m-R_m(v)$ belong to $\mathscr{N}(n)$ and $\mathscr{N}(m)$, respectively, so that A'' is equivalent to

$$A''' = (I_n-C_n(u)A''(I_m-R_m(v)).$$

This matrix has the form

$$A''' = \begin{bmatrix} \mathscr{D}_1(A) & 0 \\ 0 & A_1 \end{bmatrix} (= \operatorname{diag}[\mathscr{D}_1(A), A_1]),$$

where $A_1\in\mathscr{M}(n_1, m_1)$ $(n=1+n_1, m=1+m_1)$. Note that $\mathscr{D}_1(A)$, which divides A', also divides A'' (see the explicit form of A'') and therefore will divide A''' as well. We conclude that A is ω_r-equivalent to $\operatorname{diag}[\mathscr{D}_1(A), A_1]$, and $\mathscr{D}_1(A)|A_1$.

Now apply the same argument to A_1 in place of A, and continue this procedure r times. Recalling the last remark in Subsection 2 we conclude that A is ω_r-equivalent to a matrix of the form

$$A^{(r)} = \operatorname{diag}(\delta_1, \delta_2, \dots, \delta_r, A_r), \tag{3.10}$$

where $A_r\in\mathscr{M}(n_r, m_r)$ $(r+n_r=n, r+m_r=m)$, and

$$\delta_1\,|\,\delta_2\,|\dots|\,\delta_r\,|\,A_r, \quad \text{each } \delta_k \text{ inner or zero.} \tag{3.11}$$

The concluding arguments are essentially the same as in [3], p.308. By (3.6), ω_r-equivalence of A and $A^{(r)}$ implies

$$\mathscr{D}_k(A)\,|\,\varphi_k\cdot\mathscr{D}_k(A^{(r)}), \quad \mathscr{D}_k(A^{(r)})\,|\,\psi_k\cdot\mathscr{D}_k(A), \quad \varphi_k, \psi_k \text{ prime to } \omega_r.$$

Since φ_k is then prime to $\mathscr{D}_k(A)$ for $k=1, \ldots, r$, we infer $\mathscr{D}_k(A) \mid \mathscr{D}_k(A^{(r)})$, and hence $\mathscr{D}_k(A^{(r)})=\alpha_k \cdot \mathscr{D}_k(A)$ with α_k inner, $k=1, \ldots, r$. Thus $\alpha_k \cdot \mathscr{D}_k(A) \mid \psi_k \cdot \mathscr{D}_k(A)$, and therefore, $\alpha_k \mid \psi_k$ whenever $\mathscr{D}_k(A) \neq 0$.

Let j denote the largest among the integers $k=1, 2, \ldots, r$ for which $\mathscr{D}_k(A)$ is non-zero. Then we have for $k=1, \ldots, j$:

$$\mathscr{D}_k(A^{(r)}) = \alpha_k \cdot \mathscr{D}_k(A), \quad \alpha_k \text{ inner}, \quad \alpha_k \wedge \omega_r = 1, \tag{3.12}$$

and hence,

$$\alpha_{k-1} \cdot \mathscr{D}_{k-1}(A) \mid \alpha_k \cdot \mathscr{D}_k(A), \quad \text{with} \quad \alpha_0 = 1.$$

Now, α_{k-1} is prime to $\mathscr{D}_k(A)$ so we infer $\alpha_{k-1} \mid \alpha_k$, i.e. α_k/α_{k-1} is inner. From (3.12) we have

$$\mathscr{E}_k(A^{(r)}) = (\alpha_k/\alpha_{k-1}) \mathscr{E}_k(A) \quad (k = 1, \ldots, j). \tag{3.13}$$

On the other hand, it readily follows from (3.10) and (3.11) that $\mathscr{E}_k(A^{(r)})=\delta_k$ $(k=1, \ldots, r)$, and therefore, by (3.13) and (3.11),

$$(\alpha_k/\alpha_{k-1}) \mathscr{E}_k(A) \mid (\alpha_{k+1}/\alpha_k) \mathscr{E}_{k+1}(A) \quad (k = 1, \ldots, j-1). \tag{3.14}$$

Since α_{k+1} is prime to ω_r, α_{k+1}/α_k is prime to $\mathscr{E}_k(A)$. Therefore, (3.14) implies

$$\mathscr{E}_k(A) \mid \mathscr{E}_{k+1}(A)$$

for $k=1, \ldots, j-1$ (and then for all k).

Finally, combining (3.10) and (3.14) we see that

$$A^{(r)} = Z \cdot \operatorname{diag}[\mathscr{E}_1(A), \ldots, \mathscr{E}_r(A), A_r],$$

where $Z=\operatorname{diag}[\alpha_1/\alpha_0, \alpha_2/\alpha_1, \ldots, \alpha_j/\alpha_{j-1}, 1, 1, \ldots]$ (n terms); note that Z has α_j as a scalar multiple, $\alpha_j \wedge \omega_r = 1$. Also note that $(\alpha_j/\alpha_{j-1}) \mathscr{E}_j(A) = \mathscr{E}_j(A^{(r)}) = \delta_j \mid A_r$, and hence $\mathscr{E}_k(A) \mid A_r$ for $k=1, \ldots, r$.

This concludes the proof of Theorem 1.

4. Consider now the case of $A \in \mathscr{M}(n, m)$, where at least one of n, m is finite; it is no restriction of generality to suppose that m is finite and $m \leqq n \leqq \infty$.

Applying Theorem 1 with $r=m$ we obtain that A is ω-equivalent to the diagonal $n \times m$ matrix formed from the invariant factors of A. Now, this matrix does not depend on the choice of ω. Therefore, we have:

Theorem 2. *Every matrix $A \in \mathscr{M}(n, m)$, with m finite and with $m \leqq n \leqq \infty$, is quasi-equivalent to the diagonal $n \times m$ matrix*

$$\operatorname{diag}[\mathscr{E}_1(A), \ldots, \mathscr{E}_m(A)],$$

and we have $\mathscr{E}_1(A) \mid \mathscr{E}_2(A) \mid \ldots \mid \mathscr{E}_m(A)$.

4. Jordan models of operators of class $C_{\cdot 0}$

1. Let A, B be $n\times m$ matrix valued *inner* functions over H^∞, [3]) with *m finite* and *n possibly infinite*, $m\leqq n\leqq\infty$, and suppose A, B are quasi-equivalent. The condition for A, B to be inner implies that all determinant divisors are non-zero; in particular,

$$\omega=\mathscr{D}_m(A)=\mathscr{D}_m(B)$$

is a (scalar valued) inner function.

Choose Φ, $\Phi^a\in\mathscr{M}(n, n)$ and Ψ, $\Psi^a\in\mathscr{M}(m, m)$ such that

$$\text{(4.1)}\quad \Phi A=B\Psi,\quad \Phi^a\Phi=\Phi\Phi^a=\varphi I_n,\quad \Psi^a\Psi=\Psi\Psi^a=\psi I_m,\quad \varphi, \psi \text{ prime to } \omega.$$

Let $S(A)$, $S(B)$ be the operators defined on the Hilbert spaces $\mathfrak{H}(A)=H_n^2\ominus AH_m^2$, $\mathfrak{H}(B)=H_n^2\ominus BH_m^2$ [4]) by

$$S(A)u=P_{\mathfrak{H}(A)}(\chi u)\quad\text{for } u\in\mathfrak{H}(A),\qquad S(B)u=P_{\mathfrak{H}(B)}(\chi u)\quad\text{for } u\in\mathfrak{H}(B),$$

and set

$$\text{(4.2)}\qquad Xu=P_{\mathfrak{H}(B)}\Phi u\quad\text{for}\quad u\in\mathfrak{H}(A).$$

Then the operator X: $\mathfrak{H}(A)\to\mathfrak{H}(B)$ satisfies

$$\text{(4.3)}\qquad S(B)X=XS(A),$$

and is *injective*. These facts follow by the same arguments as in [8], Sec. 2, by giving the role of $\Psi^\wedge$ and $\det\Psi$ to Ψ^a and ψ, respectively.

Using the relation $\Phi A=B\Psi$ we get

$$\text{(4.4)}\qquad X\mathfrak{H}(A)=P_{\mathfrak{H}(B)}\Phi\mathfrak{H}(A)=P_{\mathfrak{H}(B)}\Phi H_n^2.$$

Since $\Phi H_n^2\supset\Phi\Phi^a H_n^2=\varphi H_n^2$, (4.4) implies

$$\text{(4.5)}\qquad X\mathfrak{H}(A)\supset P_{\mathfrak{H}(B)}(\varphi H_n^2).$$

Set now $\omega_1=\omega\cdot\varphi^\circ$, φ° being the inner factor of φ, and choose Φ_1, Ψ_1, etc., correspondingly. So we get X_1 such that

$$\text{(4.5)}_1\qquad X_1\mathfrak{H}(A)\supset P_{\mathfrak{H}(B)}(\varphi_1 H_n^2).$$

As φ_1 is prime to φ, by Beurling's theorem φH_n^2 and $\varphi_1 H_n^2$ together span H_n^2. As a result, the *ranges of X and X_1 together span* $\mathfrak{H}(B)$.

2. In some special cases (but not always, *cf.* [8], Sec. 3) we can choose X such that its range alone spans $\mathfrak{H}(B)$; i.e. that X be a *quasi-affinity*. Such is the case if $n=m(<\infty)$, or more generally, if $B=\begin{bmatrix}B_1\\0\end{bmatrix}$, where B_1 is a square matrix of order m, and 0 is the $l\times m$ zero matrix, where

$$n=m+l,\quad 0\leqq l\leqq\infty.$$

[3]) That is, A and B are isometry valued a. e. on the unit circle.

[4]) $H_n^2=H^2(E_n)$ is the Hardy—Hilbert space of E_n-vector valued analytic functions in the unit disc; and $\chi(\lambda)\equiv\lambda$.

For l finite, *cf.* [2]. The following generalization of the proof given in [2] applies to l infinite as well.

Choose Φ, Ψ to satisfy (4.1) with $\omega=\mathcal{D}_m(B)=\det B_1$ and partition the matrices Φ and Φ^a in the form

$$\Phi = \underbrace{\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}}_{n}\begin{matrix}\}m\\ \}l\end{matrix}, \quad \Phi^a = [\underbrace{\Omega_1}_{m}\ \underbrace{\Omega_2}_{l}]\}n.$$

Equations (4.1) are equivalent to the following ones:

$$(4.6)\quad \begin{cases}\Phi_1 A = B_1\Psi, \quad \Phi_2 A = 0,\\ \Omega_1\Phi_1+\Omega_2\Phi_2=\varphi I_n,\ \Phi_1\Omega_1=\varphi I_m,\ \Phi_2\Omega_2=\varphi I_l,\ \Phi_1\Omega_2=0,\ \Phi_2\Omega_1=0,\ \varphi\wedge\omega=1,\\ \Psi^a\Psi=\Psi\Psi^a=\psi I_m, \quad \psi\wedge\omega=1.\end{cases}$$

Clearly, $\Phi_2\in\mathcal{M}(l,n)$, with $\|\Phi_2\|_\infty\leqq\|\Phi\|_\infty$. Let

$$\Phi_2=\Phi_2^\circ\Phi_2'$$

be the canonical factorization of the bounded analytic function $\{E_n, E_l, \Phi_2(\lambda)\}$ into its outer factor $\{E_n, \mathfrak{F}, \Phi_2'(\lambda)\}$ and inner factor $\{\mathfrak{F}, E_l, \Phi_2^\circ(\lambda)\}$, where $\mathfrak{F}$ is some auxiliary Hilbert space; *cf.* [4], Chapter V. By taking $d=\dim\mathfrak{F}$ we can assume $\mathfrak{F}=E_d$; then

$$\Phi_2'\in\mathcal{M}(d,n) \quad\text{and}\quad \Phi_2^\circ\in\mathcal{M}(l,d).$$

As Φ_2' is outer, $\Phi_2'H_n^2$ is dense in H_d^2, and therefore $\Phi_2H_n^2=\Phi_2^\circ\Phi_2'H_n^2$ is dense in $\Phi_2^\circ H_d^2$. On the other hand we have, by (4.6), $\Phi_2H_n^2\supset\Phi_2\Omega_2H_l^2=\varphi H_l^2$. Therefore,

$$(4.7)\qquad \Phi_2^\circ H_d^2\supset\varphi^\circ H_l^2 \quad (\varphi^\circ \text{ is the inner factor of } \varphi).$$

On account of this inclusion, for every $u\in H_l^2$ there exists a $v\in H_d^2$ such that $\Phi_2^\circ v=\varphi^\circ u$; the map $u\to v$ defines an isometry $W\colon H_l^2\to H_d^2$ which intertwines the natural unilateral shifts on these spaces, i.e.

$$(4.8)\qquad S_dW=WS_l.$$

This implies that $l\leqq d$; *cf.* [8], Theorem 5/6.

The inclusion

$$(4.9)\qquad \Phi_2^\circ H_d^2=\Phi_2^\circ\overline{\Phi_2'H_n^2}=\overline{\Phi_2H_n^2}\subset H_l^2$$

shows that Φ_2° is an isometry from H_d^2 into H_l^2, which obviously intertwines S_d and S_l in the reverse order, and therefore, $d\leqq l$.

Thus $d=l$, and hence $\Phi_2'\in\mathcal{M}(l,n)$, $\Phi_2^\circ\in\mathcal{M}(l,l)$, and $\Omega_2\Phi_2^\circ\in\mathcal{M}(m,l)$. Therefore, both

$$\tilde\Phi=\begin{bmatrix}\Phi_1\\ \Phi_2'\end{bmatrix} \quad\text{and}\quad \tilde\Phi^a=[\Omega_1\ \ \Omega_2\Phi_2^\circ]$$

are in $\mathcal{M}(n, n)$. Moreover, it easily follows from (4.6) that (4.1) holds true for $\tilde{\Phi}$, $\tilde{\Phi}^a$ in place of Φ, Φ^a. Indeed, we have, e.g.

$$\Omega_1\Phi_1+\Omega_2\Phi_2^\circ\cdot\Phi_2'=\Omega_1\Phi_1+\Omega_2\Phi_2=\varphi I_n,$$

$$\Phi_2'\cdot\Omega_2\Phi_2^\circ=(\Phi_2^\circ)^*\cdot\Phi_2\Omega_2\cdot\Phi_2^\circ=(\Phi_2^\circ)^*\cdot\varphi\cdot\Phi_2^\circ=\varphi I_l,\quad \text{etc.}$$

The rest of the argument is similar to the one in [2]. We regard H_n^2 as the direct sum $H_m^2\oplus H_l^2$ and set $\mathfrak{N}=\tilde{\Phi}H_n^2+BH_m^2$. Since we have

$$\tilde{\Phi}H_n^2\supset\tilde{\Phi}\tilde{\Phi}^a H_n^2=\varphi H_n^2\supset\varphi H_m^2\oplus 0 \quad\text{and}\quad BH_m^2=B_1H_m^2\oplus\{0\}\supset(\det B_1)H_m^2\oplus\{0\},$$

and since φ is prime to $\det B_1$, it follows from Beurling's theorem that

$$\overline{\mathfrak{N}}\supset H_m^2\oplus\{0\}.$$

From the fact that $\mathfrak{N}$ includes $\tilde{\Phi}H_n^2=\{\Phi_1u\oplus\Phi_2'u\colon u\in H_n^2\}$ it now follows that $\overline{\mathfrak{N}}$ also includes $\{0\}\oplus\Phi_2'H_m^2$, and hence,

$$\overline{\mathfrak{N}}\supset\{0\}\oplus H_l^2.$$

Thus, $\overline{\mathfrak{N}}=H_n^2$.

Now, for the operator $\tilde{X}$ associated with $\tilde{\Phi}$ in the sense of (4.1) we have, by (4.4), $\tilde{X}\mathfrak{H}(A)=P_{\mathfrak{H}(B)}\mathfrak{N}$, and hence the closure of the range of $\tilde{X}$ equals $P_{\mathfrak{H}(B)}H_n^2$, i.e. $\mathfrak{H}(B)$. Thus $\tilde{X}$ is a quasi-affinity.

3. Applying Theorem 2 and the above results to the characteristic matrix function $\Theta\in\mathcal{M}(n, m)$ of a contraction T on $\mathfrak{H}$, of class $C_{\cdot 0}$, with defect indices

$$\dim[I-T^*T)^{1/2}\mathfrak{H}]^-=m,\quad \dim[(I-TT^*)^{1/2}\mathfrak{H}]^-=n,$$

where $m<\infty$ while $(m\leqq)n\leqq\infty$, and to the diagonal $n\times m$ matrix formed by $e_k=\mathcal{E}_k(\Theta)$ $(k=1, \dots, m)$, we conclude as in [8] and [2]:

Theorem 3. *The "Jordan operator" $J=S(e_m)\oplus\dots\oplus S(e_1)\oplus S_l$ on $\mathfrak{H}_J=\mathfrak{H}(e_m)\oplus\dots\oplus\mathfrak{H}(e_1)\oplus H_l^2$ $(l=n-m)$ is completely injection-similar to T. More precisely, there exist injections*

$$X:\mathfrak{H}\to\mathfrak{H}_J,\quad Y_i:\mathfrak{H}_J\to\mathfrak{H}\quad(i=1,2)$$

such that

$$JX=XT,\quad TY_i=Y_iJ\quad(i=1,2),$$

and the range of X is dense in $\mathfrak{H}_J$ while the ranges of Y_1 and Y_2 together span $\mathfrak{H}$.

The problem concerning uniqueness of the model can be dealt with as in [8].

Problems. 1. In [9], the existence of a unique quasi-similar Jordan model $\bigoplus_k S(m_k)$ (m_k inner, $m_{k+1}|m_k$, $k=1, 2, \dots$) has been proved for every contraction $T\in C_0$ with minimal function $m_T=m_1$. In the general case the relation of the functions m_k to the invariant factors of the characteristic matrix of T remains to be elucidated.

2. It also remains to be investigated under which conditions Theorem 1 can be sharpened so that quasi-equivalence is established to the diagonal matrix formed only by the invariant factors.

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